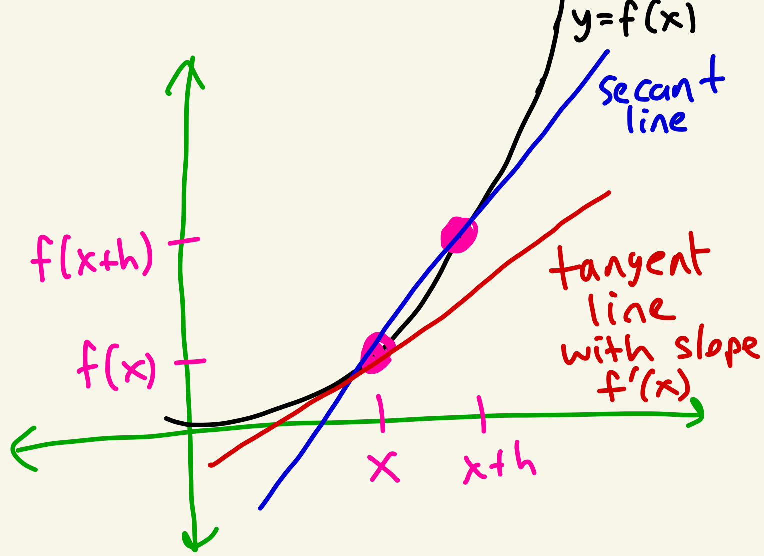
Topic 2 – Partial Derivatives

Recall from Calc I

$$f'(x) = \lim_{h \to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{slope of secant line}}$$



Def: Given $f(x,y)$, the partial derivatives of f are

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x)}{h}$$

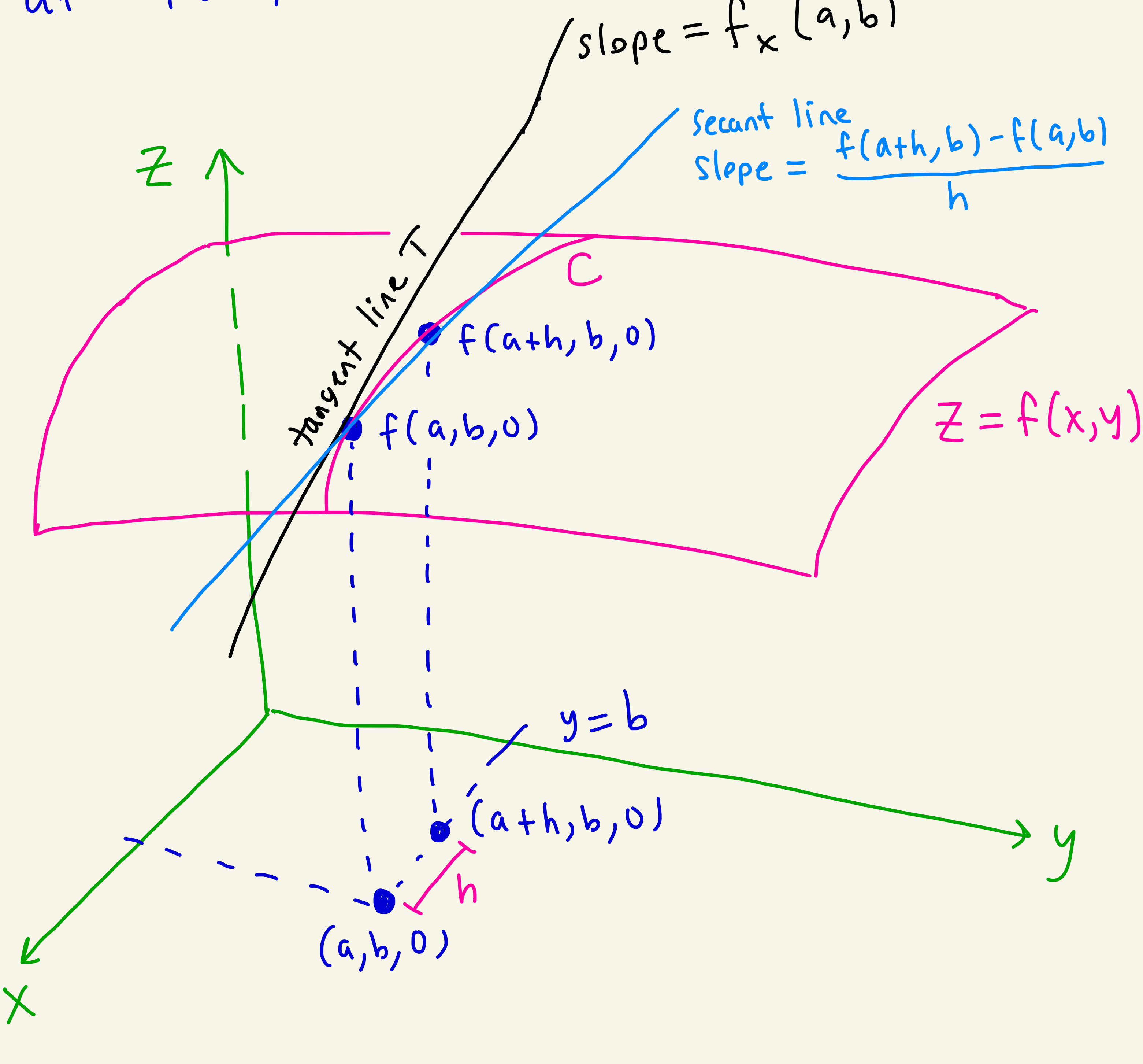
and

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(y)}{h}$$

if the limits exist.

Geometric interpretation of $f_x(a,b)$

Let C be the curve created by intersecting the plane $y=b$ with the surface $z=f(x,y)$. Then, $f_x(a,b)$ is the slope of the tangent line T to the curve C at the point $(a,b,f(a,b))$.



How to calculate f_x and f_y

- To calculate f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

- To calculate f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Ex:

$$f(x,y) = x^5y^6 + 2x^2 - y + xy$$

$$f_x(x,y) = 5x^4y^6 + 4x - 0 + y$$
$$= 5x^4y^6 + 4x + y$$

regard y as constant
differentiate f
with respect to x

$$f_y(x,y) = 6x^5y^5 + 0 - 1 + x$$
$$= 6x^5y^5 - 1 + x$$

regard x as constant
differentiate f
with respect to y

Notation: Here are some equivalent notations that people use.

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = D_y f$$

Ex: $g(x,y) = \sin(2xy^2)$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} \sin(2xy^2) = \cos(2xy^2) \cdot \frac{\partial}{\partial x}(2xy^2)$$

$$= \cos(2xy^2) \cdot 2y^2$$

$$= 2y^2 \cos(2xy^2)$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y} \sin(2xy^2) = \cos(2xy^2) \cdot \frac{\partial}{\partial y}(2xy^2)$$

$$= \cos(2xy^2) \cdot (4xy)$$

$$= 4xy \cos(2xy^2)$$

<u>Ex:</u> $f(x,y) = x e^{\cos(xy)}$

We have:

$$\frac{\partial f}{\partial x} = 1 \cdot e^{\cos(xy)} + x e^{\cos(xy)} \cdot (-\sin(xy) \cdot y)$$

$$= e^{\cos(xy)} - xy \sin(xy) e^{\cos(xy)}$$

And:

$$\frac{\partial f}{\partial y} = x e^{\cos(xy)} \cdot (-\sin(xy) \cdot x)$$

$$= -x^2 \sin(xy) e^{\cos(xy)}$$

<u>Note:</u> One can define similar partial derivatives when more variables are involved.

<u>Ex:</u> $f(x,y,z) = z^2xy + \sin(xz)$

$$\frac{\partial f}{\partial z} = 2zxy + \cos(xz)\cdot x$$

$$= 2zxy + x\cos(xz)$$

<u>Notation:</u> Given $f(x,y)$ we can differentiate f_x and f_y again to get the <u>second partial derivatives</u>:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

<u>read:</u>

f_{xy} $\longrightarrow$

$\frac{\partial^2 f}{\partial y \partial x}$ $\longleftarrow$

Ex: $f(x,y) = 5x^2y^3 - 10x^2$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= f_x = 10xy^3 - 20x \\ \frac{\partial f}{\partial y} &= f_y = 15x^2y^2 \end{aligned}}$$

first partials

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2} = 10y^3 - 20$$

$$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = 30xy^2$$

$$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = 30xy^2$$

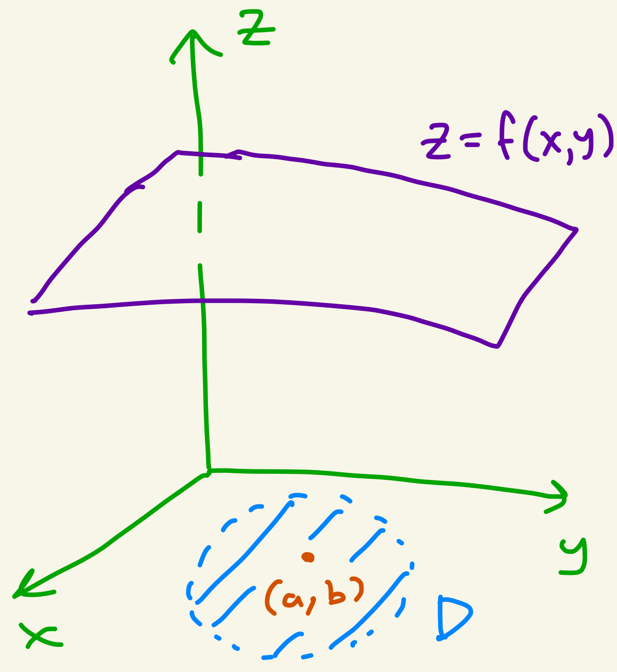
$$(f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2} = 30x^2y$$

second partials

note that $f_{xy} = f_{yx}$

Clairaut's theorem

Suppose $f(x,y)$ is defined on a open disk D that contains the point (a,b). If f_{xy} and f_{yx} both exist and are continuous on D, then $f_{xy} = f_{yx}$.



Ex: You can take higher order derivatives also.

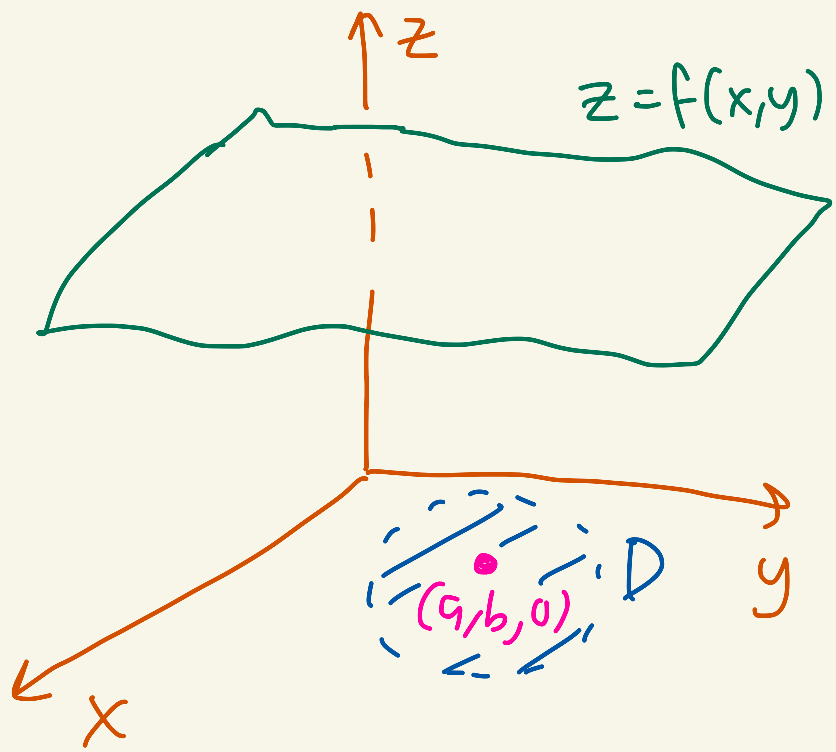
$$f(x,y) = 5x^2y^3 - 10x^2$$

$$f_x = 10xy^3 - 20x$$

$$f_{xx} = 10y^3 - 20$$

$$f_{xxy} = \frac{\partial^3 f}{\partial y \partial x \partial x} = 30y^2$$

<u>Def:</u> A function $f(x,y)$ is said to be <u>differentiable</u> at (a,b) if there exists an open disc D centered at (a,b) where f_x and f_y exist and are continuous on D.



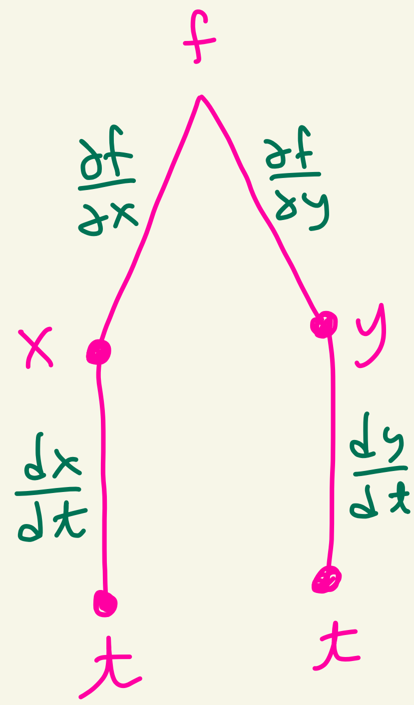
There is a more general definition of differentiable, but the above implies it.

Chain rule 1:

Let $f(x,y)$ be a differentiable function of x and y on its domain. Let $x = x(t)$, $y = y(t)$ be differentiable functions of t on their domains.

Then,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$



Ex: Let $z = x^2 - 3y^2 + 20$

where $x = 2\cos(t)$, $y = 2\sin(t)$.

Find $\frac{dz}{dt}$ and evaluate it at $t = \frac{\pi}{4}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x)(-2\sin(t)) + (-6y)(2\cos(t))$$

$$= -4x\sin(t) - 12y\cos(t)$$

$$= -4(2\cos(t))\sin(t) - 12(2\sin(t))(\cos(t))$$

$$= -32\cos(t)\sin(t)$$

$$\left.\frac{dz}{dt}\right|_{t=\pi/4} = -32\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)$$

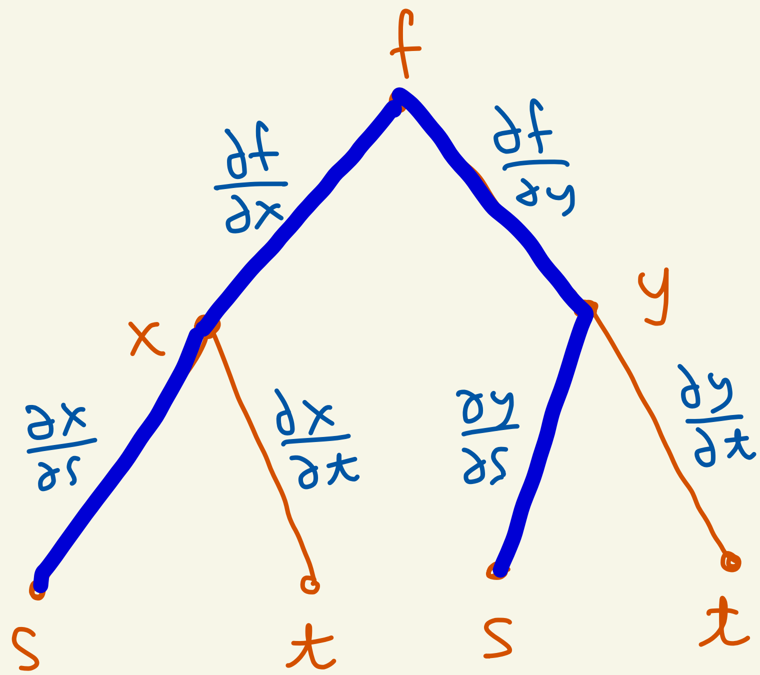
$$= -32 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = -16$$

Chain rule 2:

Let $f(x,y)$ be a differentiable function of x and y on its domain.
Let $x = x(s,t)$ and $y = y(s,t)$ be differentiable functions of s and t on their domains.

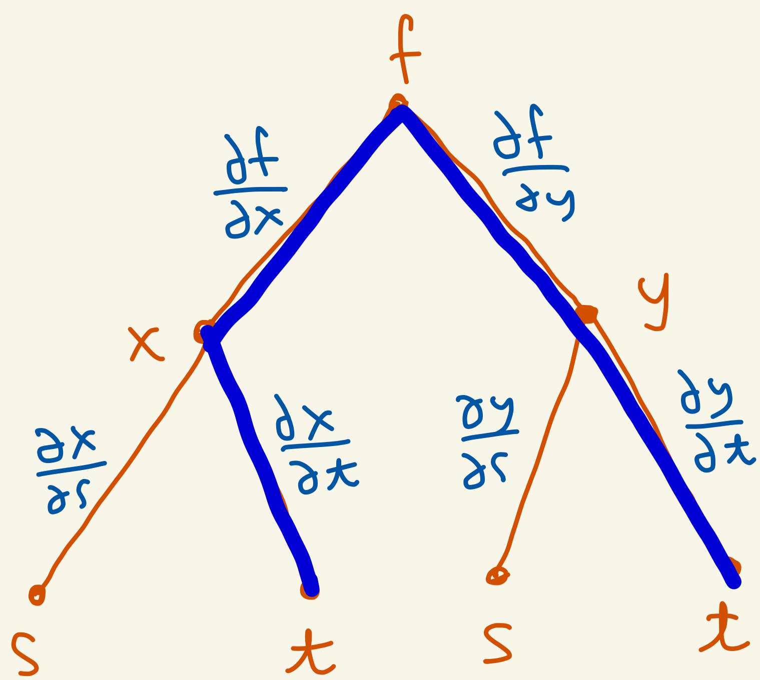
We get two formulas:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$



and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$



Ex: Let $z = e^x \sin(y)$

where $x = st^2$ and $y = s^2 t$.

Calculate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= e^x \sin(y) \cdot t^2 + e^x \cos(y) \cdot 2st$$

$$= e^{st^2} \sin(s^2 t) \cdot t^2 + e^{st^2} \cos(s^2 t) \cdot 2st$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= e^x \sin(y) \cdot 2st + e^x \cos(y) \cdot s^2$$

$$= e^{st^2} \sin(s^2 t) \cdot 2st + e^{st^2} \cos(s^2 t) \cdot s^2$$

You can create many chain rules

Ex: Suppose you have $w(x, y, z)$
and $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$

Then:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$